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ABSTRACT

This study sought to provide an explanation of (n=13) university students' understanding of the concept of linear inequality as presented in a problem-solving setting. In the course of the analysis of the data, the transition from arithmetic to algebra emerged as a critical issue. Therefore, the study examines the differences in the problem-solving activities of solvers who were able to make a transition from arithmetical methods to algebraic methods in contrast to those solvers who were unable to make such a transition. Results suggest that if solvers were to make a successful transition to algebra, they needed to attain post-representational levels of reflective abstraction. In addition, it is indicated that imagery is an inherent part of the development from one level of reflective abstraction to the next. The paper includes the learning tasks. Contains 30 references. (MKR)

Understanding Students' Transitions from Arithmetic to Algebra: A Constructivist Explanation

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UNDERSTANDING STUDENTS' TRANSITIONS FROM ARITHMETIC TO ALGEBRA: A CONSTRUCTIVIST EXPLANATION

INTRODUCTION

A review of the literature pertaining to this study revealed that the concept of variable (Kuchemann, 1978, 1981; Matz, 1979; Wagner, 1981), the simplification of algebraic expressions (Eisenberg & Dreyfus, 1988; Greeno, 1980; Rachlin, 1981; Wenger, 1987), the solution of equations (Filloy & Rojano, 1984, 1985, 1989; Kieran, 1982), the concept of rational exponents (Confrey, 1991), and the notion of inequality (Fujii, 1988) all have been the focus of research on student conceptions in algebra, although the majority of this research has not been conducted from a constructivist perspective. The research perspective is significant because it impacts the explanations given by the researchers concerning the nature of student conceptions. A fundamental difference between the perspectives is whether or not there is an objective mathematical reality that students must be made aware of and whether or not they can be expected to learn to recognize particular structures within that mathematical reality. This is a critical facet of a research perspective because it colors not only the methodology chosen and the subsequent analysis of the data but also dictates what kinds of research questions are deemed worthy of investigation and, indeed, what questions it is possible to investigate.

As any mathematics teacher knows, there are many problems that may be solved by either arithmetic or algebraic methods. The learning tasks that were used in the present study (See Table 1) are representative of these types of problems. Student successes and failures concerning such problems have long presented teachers with intriguing mysteries to solve. While expert problem solvers, familiar with algebra, might choose to write an algebraic equation or inequality to solve a problem, this path of solution is not so obvious to many novice problem solvers, including those who have

Table 1: LEARNING TASKS

TASK 1:
Horatio has decided that instead of purchasing a car, he wants to lease one. He is considering two cars. Horatio can lease a Mazda for two years for $300 per month with no additional charge for mileage. He can lease a Toyota for the same period of time for $200 per month, but there is a mileage charge of 20 cents per mile. How many miles would Horatio have to drive during the two years in order for the Mazda to be the best choice?

TASK 2:
A minivan can be rented from Buford's car rental for $ 255 per week with no extra charge for mileage. A comparable minivan can be rented from Alfonse for $189 per week, plus 22 cents per mile. If the minivan is to be rented for one week, how many miles must be driven in order for Buford's rental fee to be the best buy?

TASK 3:
You can rent a 15 foot moving truck from I-Haul rental for $100 per day plus 10 cents per mile or you can rent a comparable truck from Spyder rental for $75 per day plus 20 cents per mile. How many miles would you have to drive the truck during one moving day for it to be cheaper to rent from I-Haul?

TASK 4:
You can rent a car from "You Need Wheels" car rental for $28 per day plus 10 cents per mile or you can rent a comparable vehicle from "It's Us or a Taxi" car rental for $14 per day plus 16 cents per mile. How many miles would you have to drive the car during the one day for it to be cheaper to rent from "You Need Wheels" car rental?

TASK 5:
Two traveling businesswomen are comparing the compensation that they receive. Ann's sales performance averages 15% higher than Susan's on any given day. However, Susan receives a 12% higher commission per sale as compared to Ann's commission. When comparing the travel benefits that their companies provide them, Ann, who works for AT & Q, tells her friend that she receives $100 per day to cover her lodging and food expenses and receives 30 cents per mile she drives. Susan, who works for Spint, replies that she receives $125 per day for her lodging and food, but only 20 cents per mile driven. How many miles would Ann have to drive during one business day for her travel benefits to exceed Susan's?

Table 1: LEARNING TASKS (continued)

TASK 6:
Two traveling businessmen are comparing the travel benefits that their companies provide them. Bob, who works for a computer firm, IDN, tells his friend that he receives $100 per day to cover his lodging and food expenses and receives 30 cents per mile he drives. Ralph, who works for another computer firm, Apricot, replies that he receives $125 per day for his lodging and food, but only 20 cents per mile driven. On the business day of their conversation, which businessman received the best compensation?

TASK 7:
For it to be profitable to produce a compact economy car, the revenue raised by selling the car must exceed the costs associated with producing it. The selling price of each car is $8500. The costs associated with producing the car are $5300 per car in facilities costs (plant upkeep, electricity, etc.), $3200 per car in employee costs, and there are $500,000 of initial costs to begin production. How many cars will have to be sold in order for this product to become profitable?

TASK 8:
Review The Previous Tasks And Write Formal Symbolic Representations For Those Problems That Were Solved Using Intuitive Methods

TASK 9:
Make Up And Solve A Problem Similar To The Tasks

had considerable exposure to algebra in school. Some solvers, who are able to construct an arithmetical solution to a problem, remain unable to develop an algebraic representation. The fundamental question that must be answered concerning the problem solving activity of such solvers is– Why can the solver formulate an arithmetical solution to the problem and yet be unable to think of the problem in algebraic terms? What is getting in the way? Based on thirteen interviews and the subsequent results of five case studies (Goodson-Espy, 1994a, 1994b), this paper offers an explanation for the difficulties encountered as students attempt to make a transition from arithmetic to algebra.

Filloy and Rojano (1984, 1989) have discussed one of the difficulties students encounter in their transition from arithmetic to algebra. They defined the notion of the didactical cut to express how students' arithmetic experiences can interfere with their transition to algebra. Their term refers to the transition that occurs as students attempt to solve equations such as $Ax + B = Cx + D$. While students may be successful solving equations such as $Ax + B = C$, they are successful because such an equation may "be solved by merely undoing, one by one, the operations given in the left hand sequence, starting with the number C. We shall call this type of equation, arithmetical" (1989, p. 19). While Filloy and Rojano explained the difference between arithmetical and nonarithmetical equations, they did not precisely describe what <u>prevents</u> students from developing an understanding of the nonarithmetical equations. It is important to answer this question in order to form an better notion of what actually occurs as students attempt a transition to algebra. Nonarithmetical equations such as $Ax + B = Cx + D$ require students to move beyond an arithmetical understanding of an equation. In order to solve such equations, students must construct an understanding of an equation as two quantities that are the same. This view is necessary for understanding the properties used to solve the equation. This leads one to consider

Sfard (1991) and Sfard and Linchevksi's (1994) theory of reification and their description of the process/object duality.

In the theory of reification, Sfard (1991) and Sfard and Linchevski (1994) suggested an explanation for the problems students encounter as they attempt to develop mathematical concepts. They defined interiorization, condensation, and reification as stages in students' concept development. Interiorization was described as the stage where the learner performs operations on lower-level mathematical objects. During this stage, the learner is focused on the processes that he or she is involved with. As the learner becomes more familiar with performing these processes, he or she arrives at a point where he or she can think about what would happen without actually carrying out the process. The process is said to have been interiorized when the learner no longer has to perform the operation in order to think about the process. Condensation was described as the stage where a complicated process is condensed into a form that becomes easier to use and think about. Sfard (1991) explained the stage of condensation as being where a new concept is actually "born" (p. 19). It is not until the phase of reification that a student becomes able to conceive of a mathematical process as an object and becomes able to use this abstract object as an input for higher order processes. If, at any point this developmental cycle is interrupted, the student may resort to activity that is no longer meaningful. Such activity is referred to as being pseudostructural (Sfard and Linchevski, 1994).

In their theory of reification, Sfard and Linchevski (1994) described the process-object duality. They referred to the fact that an algebraic expression such as $3(x + 5) + 1$ may be interpreted in several different ways. They illustrated how, depending upon the context in which such an expression is encountered, the expression could be interpreted as (a) a computational process; (b) the result of computation and thus, a quantity; (c) a function; or (d) a family of functions (p. 191-192). The point of their observations is that mathematical concepts may be interpreted either operationally or

structurally and that, "mathematical objects are an outcome of reification– of our mind's eye's ability to envision the result of processes as permanent entities in their own right" (p. 194).

Sfard and Linchevski's theory of reification and their description of the process-object duality seem to have immediate application with regard to nonarithmetical equations as described by Filloy and Rojano. Students are unsuccessful with this type of equation because they view Ax + B and Cx + D as arithmetic processes only and do not consider them as quantities that are equivalent. Since they are often unable to view them as equivalent quantities, the methods taught for solving such equations can be meaningless symbolic maneuvers for the students.

In order to examine the relationship of Sfard and Linchevski's theory of reification to students' transitions from arithmetic to algebra, one must think about what allows the shifts from interiorization to condensation to reification to occur. Cifarelli's (1988) levels of reflective abstraction are very useful in specifying the details of the theory of reification. In a study that examined the role of reflective abstraction as a learning process, Cifarelli described several levels of reflective abstraction. These levels of reflective abstraction include: (a) recognition,(b) re-presentation, (c) structural abstraction and, (d) structural awareness. In relation to the present study, the level of recognition can be thought of as the ability to recognize that one can solve the current problem by doing again what one has done before. Solvers operating at this level would not be able to anticipate sources of difficulty. Solvers who are able to mentally run-through a solution and who can anticipate sources of difficulty when using previously developed methods are referred to as operating at the level of re-presentation. The next level, structural abstraction, is said to occur when a solver becomes able to mentally run-through a procedure and can reflect on potential, as well as prior activity. The highest level of reflective abstraction is structural awareness. At this level, the problem structure created by the solver has become an object of reflection.

The solver would not have to conduct mental run-throughs of solution methods in order to make judgements concerning the solution of the problem.

Sfard and Linchevski's (1994) theory and Cifarelli's (1988) levels are closely related in several respects. First, these researchers agree with the Piagetian assumption that knowledge is rooted in the activity of the learner. Next, they acknowledge the role that mental imagery plays in the development of conceptual knowledge. Finally, Cifarelli's (1988) levels of reflective abstraction may specify how a solver progresses from through the stages of interiorization, condensation, and reification. It is suggested that the levels of reflective abstraction that Cifarelli referred to as recognition and re-presentation precisely describe the cognitive activities that are required to progress through the stage of interiorization. The level of reflective abstraction referred to as structural abstraction provides the bridge from the stage of condensation to the stage of reification. Structural awareness is the level that allows the solver to attain reification. In the present study, the levels of reflective abstraction defined by Cifarelli (1988) provided a means of illustrating the theory of reification in action as solvers attempt to make their transition from arithmetic to algebra. This theoretical framework provides a basis for analyzing the activity of solvers who make a transition to algebra as well as those who are unable to do so.

PERSPECTIVE

The study was conducted from a radical constructivist perspective. The choice of this theoretical framework permitted the study to focus on the students' development of conceptual knowledge. This discussion provides a rationale for the terms and constructs used to analyze the data generated by the study and provides a context in which the results of the study can be examined.

It is necessary to define the term conception from the constructivist perspective because there are many definitions of students' conceptions based on different

theoretical constructs. The constructivist view of a student's understanding is referred to as a conception. The student's understanding of a problem may be in agreement with what is accepted in the mathematics community or it may be in conflict with that community's interpretation. In either case the student's understanding of the problem is viewed as a conception, but not as a misconception. The reason for this bias against the term misconception is that it directs attention away from the student's thought processes and toward an objective structure that the student's conception is being measured against. (Note the individual's understanding is meant from the standpoint of the observer's understanding of what the individual understood, because it is impossible to gain access to the mind of the subject in a direct way.)

The philosophic framework of research also affects the definition of problem solving that is accepted. Lester (1978) suggested the following definition of problem solving: A "problem is a situation in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the methods of solution" (p. 54). The important distinction of Lester's definition is that he emphasizes the problem from the solver's perspective while other definitions of problem solving do not acknowledge the possibility that the solver and the researcher may not be considering the same problem. This factor is crucial in determining the kinds of questions that can be asked in research studies on problem solving. For example, studies that have manipulated algebra word problems from a structural standpoint have not considered the possibility that the solvers were not concerned with the underlying structure that an expert would be aware of but instead had focused on other characteristics of the problems. The power of Lester's (1978) definition is that it allows for the study of problem solving as the problem is interpreted by the solver. This emphasis on the individual's viewpoint provides for research to be conducted that examines more than an individual's ability to solve a problem. It

provides a justification for studying the problem-solving activity of the individual and for studying the constructs that students develop through this activity.

Because this study deals with algebraic problems, it is particularly important that the notion of structure operating in the study be clarified. Based on constructivist learning principles, Cifarelli (1988) described structure as an organization of the activity that the learner performs on objects (p. 38). Structure is unobservable in the sense that the learner may not state a formal mathematical property to justify his or her activity but knowledge of such a property may be illustrated through observance of the learner's activity. The importance of structure to this study is that one can expect to see evidence of the learner's awareness of properties regarding linear inequality through the activities that he or she conducts while attempting to solve the problems. One would also expect to gather information regarding the solver's transition from arithmetic to algebra by closely monitoring his or her solution activity. In this way, the levels of conceptual knowledge that the subject has are expressed in the structure of his or her problem-solving activity. Structure in this study then refers to the organization of the learner's activity and not to a classification of the mathematical problem itself. This study does not a priori impose structures on the problems that are then looked for in the students' responses. This would be inconsistent with a constructivist perspective because the students decide not only how they will construct the mathematical concepts but what mathematical concepts will be constructed (Cobb & Steffe, 1983, p. 88).

Representation theory has played an important role in the research on problem solving. While there are several definitions of representation and its relationship to problem solving, the one of interest is that referred to by Yackel (1984, cited in Cifarelli, 1988). Representation is the initial structure the solver gives to a problem. Note that the structure lies within the interpretation of the solver, not the problem itself. As Cifarelli pointed out, Yackel recognized that this definition supports the notion that "a solver's construction of a problem representation indicates that conceptualization activity has

occurred" (Cifarelli, 1988, p. 32). Thus the representation provides the subject with an initial structure that, through the process of reflective abstraction, is modified to create more abstract structures. For the purposes of this study, representations are referred to as initial interpretations so as to distinguish them from other definitions of representation. These initial interpretations establish the structure that will provide data concerning the conceptual knowledge that students have and develop during the transition from arithmetic to algebra. Such initial structures will also clarify our understanding of students' interpretations of linear inequality embedded in a problem-solving setting.

It is crucial in this section to define the differences between initial interpretations and re-presentations. In this study as well as in Cifarelli's (1988), it is necessary to explain the initial interpretations in terms of the actions that they are coordinated with. Thompson (1985) referred to re-presentation as a means of accomplishing this. Cifarelli (1988) explained that re-presentation refers to the learner's ability to rely upon structures of activity that allow the learner to "call-up" these structures and use them to mentally model the process of solving a problem. He pointed to Mandler's (1983) work that implied individuals are not only able to develop structures of activity, they may also become able to operate on the activity in thought. The ability of the subject to operate on these re-presentations as one would operate using an object is consistent with the Piagetian notion of reflective abstraction in a problem solving setting. It is also consistent with von Glasersfeld's (1982) work that suggested that a subject would be able to use a re-presentation as a substitute for "actual sensory motor execution" (quoted in Cifarelli, 1988, p. 35). When re-presentation is described in the analysis, it means that the solver is able to mentally "run-through" a previously used or potential activity. At early stages re-presentation involves the ability to imagine oneself performing the activity and analyzing the results, in the later stages of abstract development however,

one may think of the results of an activity without having to visualize running through the activity itself.

The initial conceptual structure that the subject exhibits concerning the problem and the concept of linear inequality is referred to as the <u>initial interpretation</u>. A <u>re-presentation</u> is cited when there is evidence that a subject relied on the prior mental structure (initial interpretation) to use in subsequent attempts on the problem or later related problems. The re-presentation provides for an opportunity to witness the development of more complex levels of conceptual knowledge. von Glaserfeld's (1982) work supports this contention because he held that re-presentation is a major process in the development of abstract structures from representations or, as they are referred to in this study, initial interpretations (cited in Cifarelli, 1988, p. 44).

OBJECTIVES

The study sought to provide an explanation of student understanding of the concept of linear inequality as presented in a problem-solving setting (Goodson-Espy, 1994). The students were asked to solve problems that, from the expert's perspective, involved the concept of linear inequality. In the course of the analysis of the data, the transition from arithmetic to algebra emerged as a critical issue. Therefore, the study examines the differences in the problem-solving activities of solvers who were able to make a transition from arithmetical methods to algebraic methods as opposed to those solvers who were unable to make such a transition.

METHODOLOGY

Subjects

The subjects selected were students from a small southern university. The university's students derive mainly from three southern states with largely suburban and rural backgrounds. A total of thirteen students participated in the study. A student was eliminated from becoming a subject if that student had very recently studied the

topic of linear inequalities since it was believed that very recent exposure to the material would interfere with a subject's ability to react spontaneously to the tasks.

Use of Interviewing Methodology

Unstructured interviews were utilized because this methodology allowed me to collect data concerning the constructive behavior of subjects as they attempted to solve challenging situations that could be solved using either arithmetical or algebraic methods. I was able to ask the subjects to elaborate on their activities and strategies and made judgments concerning the students' internal problem solving processes (Opper, 1975). A key factor in enabling me to make inferences about the subjects' cognitive processes was the careful selection of the problems to be used as tasks in the interview.

Lincoln and Guba (1985) stated that "the unstructured interview is the mode of choice when the interviewer does not know what he or she doesn't know and must therefore rely on the respondent to tell him or her" (p. 269). An unstructured interview allows a researcher to have a general plan of questions or problems to be considered but the researcher is free to change the direction of the interview based on the responses of the subject. This provides for a broader opportunity to collect relevant data. One of the most critical benefits of the unstructured interview approach has been explained by Cobb (1986) as the process of "negotiating meaning." This indicates that the researcher is able to ask the subject to clarify or explain his or her activity or verbalizations, whenever there is a question as to the purpose of an activity or comment. Although there is more intervention, the intervention acts as a means of clarifying the subject's responses not as a means of directing them per se. The analysis of the data also accounts for the intervention by considering the data in light of a dialogue between the researcher and the subject. The analysis of the data must take into account questions concerning the validity of the data. Critics of the "think aloud" approach to problem solving such as Ericsson and Simon (1980) contended that if subjects are asked to

verbalize information that they otherwise would not during the process of problem solving, that this intervention would alter their problem-solving behavior and thus the data collected in this manner would be invalid. This criticism, while probably at least partially true, does not take into account the alternatives of the "think aloud" approach. If one cannot depend on the subjects' verbalizations concerning what they are doing, then one must resort to written evidence as the only source of what occurs during problem solving. Because many students write very cryptic statements while they are engaged in problem solving, it is unlikely that much could be learned through this process alone. In addition, the researcher must then attempt to interpret what the solver was doing, a task itself subject to serious questions concerning validity.

The study utilized a set of nine learning tasks (See Table 1). From an expert viewpoint, the tasks all involve the concept of linear inequality. One would not expect that all students would have this viewpoint. Instead, it was theorized that some students might approach the problems by developing an arithmetical approach based on a situation-specific imagery. Because the tasks deal with familiar situations, such as car rentals and comparisons of employee benefits, it was expected that the solver might approach the problem from a real-world perspective, interpreting the situation described in the task based on a situation that he or she had experienced.

The learning tasks must require the subject to become "task-involved" (Cobb, 1986) during the interview process. This means that the problems must be true challenges for the student, not predictable sorts of tasks that they regard as exercises to be solved via the use of some rule that they have memorized previously. The reasons for the selection of these tasks and their arrangement is the following: Cifarelli's (1988) study used a set of algebra problems that were based on the idea of simultaneous equations as a vehicle for studying the problem-solving behavior of students. Specifically, he was interested in developing an understanding of how students construct conceptual knowledge in problem-solving situations. The role of reflective

abstraction was of particular interest. His results indicated six levels of progressively abstract solution activity referred to as levels of reflective abstraction: (a) instrumental, (b) recognition, (c) reflection on a perceptual expression of a re-presentation, (d) re-presentation, (e) structural abstraction, and (f) structural awareness. Cifarelli's learning tasks derived from those developed by Yackel (1984) and were intended to investigate representation processes in mathematical problem solving. Thus, there is a precedent for using these types of tasks to gather data about student internal processes.

Data Generation

During individual interviews, the students were allowed an unlimited amount of time to solve the problems and were not told that the problems involved linear inequality. They were told that the study concerned how students solve mathematics problems. The interview typically lasted two hours and was videotaped. The data included the videotapes, transcripts of the tapes, and the written work of the students.

The need for videotaping has been established in previous research (Krutetskii, 1976; Larkin, 1985; Schoenfeld, 1985). Videotaping preserves many qualitative, nonverbal cues that written protocols alone would miss. Activities such as staring, gesturing, and talking to oneself all may occur during problem solving that written protocols would omit. In addition, the videotaped record maintains the timing and sequencing of problem solving activity. Another benefit provided by videotaping is that it allows the interviewer to become an observer and reevaluate the interview from this less involved perspective. This is extremely useful because the repeated viewing of the tape during analysis allows the interviewer turned observer to continually search the data for important information and to alter the model to more accurately depict the data.

The written protocols are also extremely important. The written protocols include the transcripts of the videotapes and the paper records made by the subjects as they

solved the problems during the interview. The written transcripts are numbered so that the subjects' activity may be easily referred to during the analysis of the data (Goodson-Espy, 1994). The records made by the subjects themselves are, of course, very valuable because these data provide clarification for some comments and actions noted on the tape. For example, some of the solvers referred to charts and it is crucial to have those data available.

Analysis of Data

The written and video protocols for each subject were analyzed and the results presented in case studies. An explanation of students' transitions from arithmetic to algebra was described based on these results. In addition, the findings include a characterization of students' conceptions of linear inequality.

The tasks were selected with the intention of providing a genuine challenge for the subject with the hope that cognitive conflict would occur and allow for the development of conceptual knowledge. I observed and recorded, through written and videotaped protocols, the ways that reflective abstraction affected the development of conceptual knowledge during the transition from arithmetic to algebra.

The case studies were prepared initially by a narrative concerning each subject's activity while solving each task. An important aspect of this stage was defining the meaning that the subject intended as he or she attempted to solve the problem. The narrative organized both cognitive and metacognitive data concerning each subject. The collection of metacognitive data occurred as the subject and I negotiated meaning about the subject's activities and responses. Data revealing qualitative aspects of the student's behavior were recorded on videotape.

After the initial narrative was prepared for a solver, a summary of the solver's constructive activity was developed. Finally, the solver's performance was examined for unique activity that revealed how the solver thought during the transition from

arithmetic to algebra (if such a transition took place). The data were also analyzed to form an impression of the solver's understanding of linear inequality.

Cifarelli's (1988) levels of reflective abstraction were used as a guide to classifying the problem-solving activity of the subject. This reliance on reflective abstraction and the processes of re-presentation can be justified through the research of von Glasersfeld (1982, cited in Cifarelli, 1988), who found that these processes were significant in the development of structural knowledge. The analysis and classification of the problem-solving activity yielded valuable information about the subjects' levels of conceptual knowledge during the transition from arithmetic to algebra. The analysis also provided a means for characterizing the solvers' understandings of the concept of linear inequality embedded in a problem-solving setting. Based on the data, an explanation of students' transitions from arithmetic to algebra is described and supported through the citation of examples of subject activity contained in the case studies.

FINDINGS

The solvers' solution activity concerning Task 1 led to the development of a conceptual structure referred to as the initial interpretation. Based on their beliefs that a problem was similar to one that they had solved before, the solvers used or modified this initial interpretation. The solvers made decisions concerning the appropriateness of prior solution methods as well as potential methods. Solvers operating at higher levels of reflective abstraction were able to reflect on prior and potential solution activity and make anticipations concerning the results without having to physically carry out the procedures. The analysis of the case studies fell into three categories: (a) students who used purely arithmetical methods that were not based on situation-specific imagery, (b) students who utilized charting methods that were based on situation-specific imagery and, (c) students who used formal algebraic methods. The following

discussion will include episodes from the case studies of solvers who were typical of each category and the levels of reflective abstraction that were attained will be noted. Finally, the levels of reflective abstraction that a solver was able to attain will be compared to that solver's success or inability to make a transition from arithmetic to algebra.

First, while Solver #11 used an arithmetical method that was based on situation-specific imagery to solve Task 1, her work on subsequent tasks was ordered by the process she had defined to solve Task 1. She was unable to successfully modify her method to account for the changes present in the other tasks and made arithmetical mistakes that became a part of her interpretation of the task. As a result of this, her work ceased to be based on situation-specific imagery. Furthermore, her work on Task 8 reveals that the solver attempted to use the initial interpretation she had constructed for Task 1 as a guide for developing an algebraic formula that could be used to solve all of the tasks.

Solver #11 completed tasks 1-9 during an interview that lasted 110 minutes. As the following episode will illustrate, the solver's initial interpretation of the solution to Task 1 was an arithmetic method that was based on situation-specific imagery. After a brief pause that followed her reading of Task 1, she began working on the task by recording the information that she was given and then she brought up the need to account for the time of the lease, "I'm going to go ahead and multiply this by 12 so I'll know how much it's going to be a year." She proceeded to multiply both of the rental charges by 12 to obtain the amounts $3,600 and $2,400. At this point she mentally calculated the difference between 3,600 and 2,400 as evidenced by her discussion concerning what to do with the $1,200:

> S: O.K., um, 20 cents per mile, let's see, 20 cents, 12 months anyways, per mile. In order for this one to be the best choice, this will have to equal $1,200. And so I have to put the 20 cents into the $1,200.

The solver was reluctant to use a calculator and chose to finish her calculations in the following manner: "O.K., so I'm going to multiply 5 times 1,200 because that's how many times it'll go into it." This computation produced the answer 6,000 and the solver announced that, "He'd have to drive it 6,000 miles." Solver #11 seemed to have a reasonably good grasp of the situation described in the problem with the exception of understanding the duration of the lease. Because she had calculated her answer based on half of the lease period, her answer was half that of the correct answer, 12,000 miles. However, the solver did not approach her calculations in a rote way. As the above comments attest, she provided explanations for why she felt that what she was doing was appropriate. The solver also seemed to have a good understanding of equivalent arithmetic operations because she was aware that if she multiplied 1,200 by 5 she would obtain the same answer as dividing 1,200 by .20. In short, the solver did not seem to have difficulty understanding the problem described in the problem and her failure to consider a 2-year lease rather than a 1-year lease was probably merely an oversight.

Unfortunately, the solver's insight that multiplying by 5 would be equivalent to dividing by .20 led her to make an error as she solved the second task. Since the solver regarded the second task as being similar to the first task, she decided to use the same method of solution. After finding the difference between the rental fees, she multiplied this difference by the mileage charge. At this point, the solver's interpretation of the problem became closely tied to the process of subtraction followed by multiplication. The significance of this process to the solver can be established by noting that during her work on Task 3, she was unable to develop meaning for the occurrence of a second mileage charge.

> S: I-Haul $100 a day and 10 cents per mile and Spyder $75 a day and 20 cents per mile. I want to make it cheaper to rent from this one, O.K. So the difference is $25 a day. (pause) Um, so I'll multiply it times 20 and get 500. You can drive it 500 miles to make it cheaper.

> I: O.K. You chose to multiply by what?

S: I chose to multiply the difference of these two [the two rental charges] by the number of cents per mile on that one [Spyder truck].

Thus, she did not modify her solution activity and her interpretation of the problem ceased to be based on situation-specific imagery. On the remaining tasks, the solver attempted to use this arithmetic process and her solution activity remained detached from situation-specific imagery.

Solver #11 seemed concerned about whether she would be able to perform well concerning Task 8 until, during my explanation of the instructions, she exclaimed, "Write a formula!" Once the solver determined that she needed to write a formula, she seemed more comfortable and confident. The solver proceeded by writing what she referred to as an outline for her formula: "Subtract the two prices, multiply the difference by the rate." She then began to describe her formula in terms of variables.

a = price #1
b = price #2
c = difference
d = rate
a - b = c x d

The solver was asked to solve Task 3 by using the formula that she had defined. She realized that she did not know which rate to use in her formula and changed the formula to "d = rate of lower price". Once the solver had made this modification, she offered this formula as a means of representing all of the problems algebraically.

Solver #11's activities on all of the tasks indicate that she did not advance past the level of reflective abstraction referred to as recognition. While she was able to identify sor : similarities between the problems, she was unable to anticipate sources of dif. :ulty and did not engage in mental run-throughs of the problems. Her initial interpretation of Task 1 served as the process that guided her through the remaining problems. The solver was not able to modify her interpretation to accommodate new information. Not only was the solver unable to anticipate sources of difficulty, she

chose to ignore aspects of the tasks that would not work in the computational procedure that she had developed.

The solver's work concerning task 8 is interesting for several reasons. First, her formula, a - b = c x d, reveals that she is not regarding the equals sign as saying that two quantities are the same, instead it only serves to announce the answer of the first computation, which was to be used in subsequent computations. Next, her work on this task indicates that she is using letters as labels and that she is not thinking of the changing mileage as a variable quantity. Finally, her work is intriguing because it emphasizes how students can think of writing algebraic representations in terms of writing formulas, yet for the student the meaning of the word, formula, may not necessarily be the same as that notion held by an expert. For this solver, writing a formula was a direct attempt to code her arithmetic processes. Following her solution method, the solver realized that she had first subtracted and then multiplied. Recall that she multiplied because she had discovered in her work on the first tasks that multiplying by 5 was equivalent to dividing by .20. While writing an equation such as 7,200 = 4,800 + .20x does require one to subtract and then divide in order to solve for the variable, it is not helpful to think about those processes when one wants to write an equation to represent the problem. It seems that she was unable to think of her algebraic representation as a way of showing a relationship between quantities, for her, the formula was a only a means of formalizing her arithmetic processes. The following section describes the solution activity of Solver #4, who like Solver #11 used an arithmetical method, but unlike Solver #11 was able to keep her solution activity grounded in the situations that were described in the tasks.

The work of Solver #4 illustrates the work of a solver who used situation-specific imagery to develop arithmetical solutions to the tasks but who was unable to construct algebraic representations. Her interview lasted 95 minutes. She used an arithmetic

method to solve the first two tasks. This arithmetic method was based on situation-specific imagery and served as the solver's initial interpretation of Task 1.

> S: O.K. He's considering two cars. He can lease a Mazda for two years for $300 per month at no additional charge for miles and he can lease a Toyota for the same period of time, two years, for $200 a month but there is a mileage charge of 20 cents per mile. How many miles would Horatio have to drive during two years...O.K.. If this is going to be $300 per month for two years, that would be $3600 a year. O.K., so for two years that would be $7200 and the Mazda is $200 per month which would be $4800 plus he's got to pay 20 cents per mile. Alright, so I can look at the difference between the two as far as total cost and then determine how many miles that's going to be at 20 cents per mile. ... O.K., I think you would have to go 12,000 miles, to go over 12,000 miles.

> I: How did you arrive at that?

> S: Um, well, let's see...O.K. if he drives 12,000 miles that would be $2,400. I divided the 20 cents into the $2,400.

Obviously, this solver possessed a thorough understanding of the nature of the problem and proceeded to implement a completely logical arithmetic solution which she repeated to solve the second task. The solver's work on Task 2 reveals that she was operating at the level of recognition. On the third task, which includes an additional mileage charge, the solver began to set up a chart to organize what she was doing. She realized that there was a $25 difference in the rental fees and set about comparing the mileage charges.

	I-Haul $100/day .10/mile	Spyder $75/day .20/mile
50 miles	.10(50) = $5	.20(50) = $10
100 miles	.10(100) = $10	.20(100) = $20
200 miles	.10(200) = $20	.20(200) = $40
250 miles	.10(250) = $25	.20(250) = $50

Figure 1. Task 3 chart

The solver repeated her charting method for Tasks 4 and 5. During the solver's work on the remaining tasks, she worked mainly at the level of recognition. Only on Tasks 6 and 7 did the solver attain the level of re-presentation. The evidence that she is operating at the level of re-presentation centers on the fact that she did not develop a chart to solve

Task 6, instead, she was able to mentally run-through what would happen if she applied the chart strategy and was aware that she would not be able to answer the question. The solver's response to this task illustrated her ability to relate a new problem to the structure that she has already created. She was able to recognize the similarities between the problems but she was also able to discern the differences. In addition, it revealed her capability to reflect on the novel aspects of a situation to devise means of coping with missing information. The solver was able to operate at the level of re-presentation on Task 6 because she was able to anticipate the difficulties she would encounter without actually having to construct a chart to solve the problem.

Solver #4 began to experience difficulty during task 8 when she was asked to solve task 1 by using algebraic methods.

S: I could let...I suppose I could let x represent car number one [Mazda]. ...It's terrible that I don't even really know where to begin. I'm not sure I can do this. I'm trying to make a distinction between the two different cars. x should represent what I'm looking for...is the difference in mileage. Or the amount of miles that one has to drive for one [Mazda] to be the better deal. Wait a minute...[Writes 300x and 200x + mileage charge] I'm not seeing through it, it's so much easier to do it the other way. I'm trying to put it into symbols in my mind and it's just not coming together.

I: Let me ask a question. Where you have 300x and 200x plus the mileage charge there, why are you multiplying the 300 times x and 200 times x? What does that mean?

S: Well, that's bad. Um, I was trying to decide if x could represent some one thing and I could set this up as one equation but I don't see how to do that.

I: O.K. And so x was representing what?

S: It was– I was trying to decide. I was going to make the x represent what I'm looking for, the amount of miles that would make the one car [Mazda] a better deal, but in so doing, I still do not see how to set it up as an equation for the whole problem.

The solver's initial interpretation of an algebraic representation was an effort to represent "what we're trying to solve for." It is significant that Solver #4 did not attempt

to code the arithmetic processes that she had used on the first two tasks. While she was not able to produce an algebraic representation that satisfied her. it is worth noting that she was aware that attempting to code her arithmetic processes would not be a productive path. This solver's work is also intriguing because it begs the question– If she can construct arithmetical solutions to the tasks, what prevents her from constructing algebraic solutions? The solver's use of a chart on Task 3 and on subsequent tasks serves as evidence that the solver was capable of using symbolism. She was able to identify some characteristics of an algebraic form, as indicated by her insight of x as "what we're trying to solve for." This does not indicate that this notion represented a variable quantity for her, however. She seemed to understand the need to assign "what we're trying to solve for," to some symbolism, but did not appreciate how that symbolism would relate to the structure of the problem as she had solved it by using her chart. While the solver was able to use the chart as a means for finding out when the mileage charge on one vehicle would equal the difference between the rental fees, it seems that the chart only served as a method for organizing her arithmetical processes. She did not seem able to think of the chart as representing quantities that she was comparing and thus she could not think about how the chart could be used to express those quantities through the use of a variable. The activity required to construct an algebraic statement was a source of great difficulty that this solver was unable to overcome.

One explanation for this solver's dilemma can be found by examining the levels of reflective abstraction that the solver attained during her problem solving. During her work on the tasks, the solver worked mainly at the level of recognition. Only, briefly, on Tasks 6 and 7 did the solver attain the level of re-presentation. Thus, this solver's solution activity was restricted to the lower levels of reflective abstraction. It may be useful to compare the levels of reflective abstraction that this solver attained with the levels attained by the solvers of the following case studies who made progress in the

transition from arithmetic to algebra. As will be noted in these episodes from the case studies, some solvers who were able to represent solutions of the tasks by using charts were able to extend their use of symbolisms to include algebraic symbolisms. This did not occur in the case of Solver #4 and suggests that the ability to use charts does not automatically infer that a solver will be able to cross the chasm that lies between intuitive symbolisms such as charting and more formal algebraic symbolisms.

Solver #6 was another student who used charts to help her solve some of the tasks. The solver's interview lasted 110 minutes. She found answers to all of the tasks by using arithmetical methods and charts that were grounded in situation-specific imagery. During her work on Task 8, she was able to use a variable to represent the unknown mileage on Task 1 and other tasks but she was unable to complete a transition from arithmetic to algebra. The following episode illustrates the arithmetic method Solver #6 used to solve the first task. This arithmetic method served as the solver's initial interpretation.

> S: There are 12 months in a year and 12 times 2 is 24. 24 • 300 = $7,200 and 24 • 200 = $4,800 so we have to find out, let's see, 7,200 - 4,800 = 2,400. O.K. and .20 will go into 2400. 12,000.

The solver operated at the level of recognition and used the same method to solve the second task. She tried to use the method to solve the third task but ran into trouble because she couldn't decide what to do with the additional mileage charge. On the fourth task, the solver developed a chart to help her solve the problem.

> S: I'll just start with 100. (uses calculator)
> 100 • .10 = 10   28 + 10 = 38
> 100 • .10 = 16   14 + 16 = 30
> It's getting closer, it should be, it had to be. Let's try 200 miles, I know there's some algebraic form for this but I don't know. Let's see, (uses calculator)
> 200 • .10 = 20   28 + 20 = 48
> 200 • 16 = 32   14 + 32 = 46
> I'm getting closer. Let's try 250 miles. (use calculator)
> 250 • .10 = 25   28 + 25 = 53
> 250 • .16 = 40   14 + 40 = 64
> Aha! I don't have to hope. O.K. I'll try one more time at 225 and see if that's it, a little bit closer.

225 • .10 = 22.50   28 = 22.50 = 50.50
225 • .16 = 36   14 + 36 = 50
Whew, I guess, I bet he has to drive 226, no 227, no 226. This better be the right one.
226 • 10 = 22.60   28 + 22.60 = 50.60
226 • .16 = 36.16   14 + 36.16 = 50.16
No, well, 230 miles. This is taking way too long.
230 • .10 = 23.00
230 • .16 =
What's the difference? Pretty big difference. It's only a difference of $2.00. (re-reads the task) How many miles does he have to drive the car during one day for it to be cheaper to rent from "You Need Wheels"? O.K. Somewhere between 250 and 225.

The solver worked at the level of recognition and used her charting method to find a solution for task 5 and decided to redo task 3 using her new method. The solver attained the level of re-presentation with respect to Task 6 since she could mentally run-through what would happen if she tried to use a chart.

She provided algebraic representations, with difficulty, when she was asked to do so on Task 8. On this task she worked at the level of recognition with respect to the algebraic representations of the tasks she had previously solved using an arithmetic or charting method. She wrote, x = [(300 • 24) - (200 • 24)] ÷ .20, as her symbolic representation for her work concerning the first task, referring to it as, "my algebraic!". In her representation for the second task which follows, she altered her use of the letter x, allowing it to represent the number of miles to be traveled.

S: x = Now I can't write an algebraic equation for this one. . . maybe. . . (x • .22). Well, maybe I can. (x • .22) . . . No, I can't, wait, no. . . (x • .22) + 189 =, O.K. there's my algebraic equation. Equals? Um, I don't know, I can't do that one.

While the solver used charts to solve some of the problems, because she was operating at lower levels of reflective abstraction, she was unable to think of the processes used to create the charts as abstract objects. The following episodes illustrate this:

Task 3
S: I'm comparing the two [costs for renting the vehicles] and I don't see how I could write an algebraic equation to compare them. Cause I mean, I'm sure I

could write down the algebraic equation easily like, (X • .20) + 75 and then (X • .10) +100, but then I would have to compare them.

I: Well, what was the comparison? What was the question asking? What are you doing at each stage of the chart?

S: I have to compare and I don't know.

Task 4

S: (x • .10) + 28 and (x • .16) + 14, I can write it, I just can't use it!

These examples illustrate that for Solver #6, a phrase such as (x • .10) + 28, exists only as an arithmetic process. She was unable to think of it as the total cost of renting a truck and, lacking the ability to view this phrase as an abstract object, she literally had no thing that she could compare to something else. While she referred to (x • .20) + 75 and (x • .10) + 100 as an equation, she never wrote an equals symbol between them and her comments reveal that she did not consider doing so.

Solver #6 could not complete a transition to algebra because she was working with a statement such as (X • .20) + 75 as a process only. She was unable to conceive of it as a quantity that could be compared to another quantity. This incapacity to cope with the process/object duality as described by Sfard (1991) was a result of Solver #6 being unable to attain levels of reflective abstraction higher than re-presentation. In terms of Sfard's (1991) constructs, the solver remained at the level of interiorization. The level of structural abstraction was required to enter the phase of condensation and the solver was unable to reach this level. The solver needed to pass through the phases of condensation and reification in order to develop the ability to consider her arithmetic processes as objects. Furthermore, because the solver remained at the stage of interiorization, she held process-oriented conceptions of variable and equality.

It is interesting at this point to compare the activities of Solver #4 and Solver #6 and to offer some explanations for why Solver #6 could develop some algebraic symbolism while Solver #4 remained unable to do so. Solver #4 was not aware of the use of variable that was present in her charts. While she used the charts to organize her arithmetic activity, and while she allowed the mileage to change in her charts, she never

viewed this changing mileage as a variable quantity. She was unable to express this idea in words and was unable to use a variable to express the process used to create the charts. Solver #6, on the other hand, was a aware of the process that she was using and was able to think of the changing mileage as a variable. She was able to verbally and symbolically express this relationship. She was unable, however, to think of the results of that process and thus the process itself as a quantity or object.

Solver #12 completed the tasks in an interview that lasted 80 minutes. Solver #12 was able to solve the first two tasks by using an arithmetical method that was based on situation-specific imagery. Like the other solvers, he solved this task by finding the difference between the lease charges, $2,400 and dividing this difference by the mileage charge, .20. This arithmetical method was based on his initial interpretation of Task 1 and he operated at the level of recognition with respect to his activity concerning Task 2.

After reading over Task 3 briefly, the solver exclaimed, "It's something different!" He read the task aloud and then stated the charges associated with both of the moving trucks and noted that the I-Haul truck was to be the best deal. At this stage he began to search for a method to solve the problem. He began trying to use the arithmetical method that he had constructed during his solution of Tasks 1 and 2 but he was puzzled concerning what he should do with the extra mileage charge. The solver finally decided to compute the difference between the rental charges (100 - 75 = 25) and then computed the difference between the mileage charges (.20 -.10 =.10). He then multiplied these numbers together. He was unhappy with the answer he obtained (2.5) however, and began to look for another method. This episode is significant because it illustrates how the solver made decisions based on what was reasonable in the context of the situation. He tried to solve this problem by adapting the method that he had used successfully on the previous tasks. It is very significant that the solver was able to adapt the method he had used on the previous tasks. This indicates greater flexibility in his thinking and is

an indicator that he is operating at the level of re-presentation. He chose to reject this method when it produced what he regarded as unreasonable results but he seemed unsure of what had gone wrong. His inability to anticipate what had gone wrong indicates that he was not operating at the level of structural abstraction. He did not seem to reflect upon this arithmetical method and favored discarding it altogether. His ability to relate his computational results to his imagery regarding the problem caused him to consider an alternative solution method. His reliance on situation-specific imagery was very significant because he had more confidence in his imagery of what should happen than in the computation procedure he had devised. He used the arithmetical method as a tool and was willing to discard it when its results no longer made sense to him.

Once the solver had determined that his method was producing unacceptable results, he decided to try another way, "You could just draw a chart and compare so if you just went $100 for the I-Haul and $75 for the Spyder and for the I-Haul everything is 10 cents a mile and for the Spyder, it's 20 cents a mile." He constructed the chart shown in Figure 2, using 50 miles as his first guess about the mileage.

	50 miles	100 miles	150 miles	200 miles	225 miles	230 miles	250 miles
I-Haul							
$100	5	10	15	20	22.50	23	25
10 cents/mi	105	110	115	120	122.50	123	125
Spyder							
$75	16	20	30	40	45	46	50
20 cents/mi	85	95	105	115	120	121	125

Figure 2. Task 3 chart.

The solver's statement regarding the answer to this task is consistent with his behavior concerning the other answers. Once he determined the mileage at which the charges were equal, in this case at 250 miles, he used this figure to help him declare the answer of 251 miles.

The solver initially worked on this task at the level of re-presentation. He was operating at least at the level of re-presentation on this task because he knew he would have to adapt what he had done on the previous tasks. He was able to anticipate having to adapt his method but was unable to anticipate the results of the adaptation. He was also unable to determine what had caused his first attempt to fail. The solver resorted to using the chart method following this failure. It is probable that he was capable of using a chart method on any of the first three tasks but did not seem to think of a charting method as being a very efficient way to solve the problem. His statement , "You could just draw a chart," suggests that he does regard this a legitimate way of proceeding, however.

Solver #12 was able to solve Task 4 very rapidly. He was working at least at the level of recognition because his activity closely resembled what he had done on Task 3. In addition, there is some evidence that learning was taking place because his initial guess about the mileage on Task 4 was much closer than it was on Task 3. The solver was beginning to anticipate the results of his computations. The solver reached the level of re-presentation concerning Tasks 5 and 6 because he was able to mentally run-through what would occur in his solution method and anticipate the results.

During his work on Task 8, the solver developed algebraic representations for problems that he had solved earlier by using arithmetic or charting methods. For the first task, the solver wrote, $300 = 200 + (.20)X$. He wrote, $255 = 189 + (.22)X$ to represent the second task. Solver #12 carefully analyzed what he had done in the chart to solve Task 3 before he wrote his algebraic representation.

> S: Um, I think you could do it like, um, put the 100 per day, add that to your 10 cents on the mile, times x, and x is your number of miles, and that's going to have to equal your 75 per day plus 20 cents also times X.
> [Writes] $100 + (.10)x = 75 + (.20)x \quad x = \text{miles}$

While the solver wrote, x = miles, his verbalizations indicate that he is thinking of it as representing the number of miles. This demonstrates that a more robust understanding

of the concept of variable is developing out of the solver's activity. After working the problem to check his solution, he decided that he wanted to change the equation to an inequality. In the two case studies where a transition to algebra occurred, the solvers first represented their work with an equality and only later translated it into an inequality. This results supports Sfard and Linchevski's (1994) observation that equality precedes inequality in conceptual development. Solver #12 changed the representation to $100 + .10X > 75 + .20X$. The reader will observe that the inequality symbol is reversed. In the transcript of the solver's charting, it was apparent that the solver was using the notion of finding a critical point and adding one. Thus, during his algebraic representation he bases the direction of the inequality on his prior activity. While this error is interesting, it does not detract from the solver's ability to make a transition to algebra.

On Task 8, the solver was able to develop algebraic representations for the tasks by mathematizing what he had done to create the charts. Through his activity, he became able to think of an arithmetic process, such as $100 + (X \bullet .30)$, as an abstract object, such as the total cost of renting a truck, that could be symbolically compared to another abstract object. Developing this capability allowed the solver to write equations and inequalities to symbolize the comparisons he had made while using his charts.

The problem-solving activity of Solver #12 was very similar to the activity of Solver #6 until Task 8. It was on Task 8 that Solver #12 began to exhibit characteristics that suggested that he was operating at levels of reflective abstraction higher than re-presentation. Whereas Solver #6 had been able to symbolically represent parts of the tasks algebraically [i.e. $(x \bullet .10) + 28$ and $(x \bullet .16) + 14$] she had been unable to finish the sentences by placing a relational symbol between the two phrases. Solver #12 was able to produce complete algebraic representations on Task 8. Solver #12's activity indicates that during the course of solving the learning tasks through charting, he had become able to think of his arithmetic processes as abstract objects that could be manipulated.

Thus, he was not only able to write algebraic phrases to represent what he had done to create the charts, he was able to write complete algebraic sentences to compare the quantities he had created. While working on Task 8, the solver was operating at the level of structural abstraction regarding algebra. He used the algebraic structure he had constructed to write algebraic equations or inequalities for all of the tasks, except for those with insufficient information or contradictions. Solver #12 was able to complete a transition to algebra because he was able to attain the levels of structural abstraction and structural awareness as he solved the problems.

The solver was able to realize higher levels of reflective abstraction by mathematizing his earlier activity. Achieving the level of structural abstraction allowed the solver to think of his arithmetical statements in terms of what they meant to him in the context of the situation-specific imagery he had used to solve the problems. This activity permitted his conceptions to undergo the condensation phase. Attaining the level of structural awareness allowed the solver to complete this phase and to think of these arithmetic processes as abstract objects that could be manipulated. This activity illustrated reification in action.

Solver #8 was the only solver who completed all of the task by using algebraic representations from the beginning. His interview lasted 40 minutes. The solver operated at the level of structural awareness with respect to arithmetic. This solver did not have to consider the problem from an arithmetic standpoint first and then reorganize and abstract from this activity as the others solvers had needed to do. He was able to begin considering the problem from an algebraic viewpoint. The solver should be characterized as operating at the level of structural abstraction with regard to algebra on this task. The solver proceed through Task 1 by underlining what he determined to be important in the problem and then write symbolic statements to represent the rental fees that were involved. By stating, "So the total cost for the Mazda is $300 and the total cost for the Toyota is $200 plus .2x, the number of miles," he is

clearly establishing that he understands that these are the quantities that he wants to compare. He also indicates that he is interested in comparing the quantities when he decides to set them equal to one another in order to fomd the break even point. It is useful to note here that even solvers who are capable of working at very high levels of abstraction tend to write equations to express relationships rather than inequalities. If an inequality form can be used, an equation will do because solving the equation will provide the needed critical point.

On Task 3, the appearance of an additional mileage charge did not prove to be a source of cognitive conflict for this solver. On the previous two problems he had already thought of one rental fee plus an additional mileage as a single quantity and had been able to represent this quantity by a symbolic expression. Now it was not difficult for him to do this for two vehicles. The solver was focused on the quantities that were to be compared, not arithmetic processes. It is not so significant that the solver was able to represent the quantities symbolically, Solver #6 had been able to do this but could go no further. What is significant is the way that Solver #8 thought about his symbolic representations. These sentences were quantities or objects for the solver in addition to being processes. This is of crucial importance because viewing these statements as objects gives one something that can be compared. It seems that one is unable to write complete symbolic representations unless reification of the process has occurred and the process can be thought of as a quantity. This reification had occurred for Solver #8 and he was able to write algebraic representations for the tasks very easily.

DISCUSSION AND CONCLUSION

This study revealed the levels of conceptual knowledge that the solvers used as they attempted to resolve problematic situations. These levels were exhibited through the solvers' problem solving activities. Detailed analysis permitted the solvers' activities to be characterized in terms of Cifarelli's (1988) levels of reflective abstraction.

Transitions between levels were identified by observing episodes where novel behavior occurred, an indication of constructive activity.

The results of all five case studies revealed that if a solver was to make a successful transition to algebra they needed to attain post-re-presentational levels of reflective abstraction. In addition, the results indicate that imagery is an inherent part of the development from one level of reflective abstraction to the next. This means that the kinds of imagery that a solver is capable of using evolves and this imagery is significant in determining the levels of reflective abstraction that the solver can attain. Highly flexible thinking is indicated when solvers are able to rely on various forms of imagery. By this I refer to the process-object duality and underscore the fact that imagery pertains not only to processes that describe problem situations but also to the objects that evolve from those processes.

While the first three solvers worked the majority of the tasks at the level of recognition, it is clear that their activity cannot be solely described in this manner because there is a great deal of variation in the ways that the solvers approached the tasks. It is believed that the effects of beliefs, expectations, and situation-specific imagery can account for some of these differences.

The first three solvers, who operated at the levels of recognition and re-presentation, typically held weak conceptions of variable and equality. A letter simply served to name a value they were looking for or it served to name an arithmetical process that they were describing. Solvers operating at these levels did not use equality sentences at all or used the equals symbol in an arithmetic sense– to announce the answer to a computation. These solvers were unable to conceive of arithmetic processes as objects and thus their transition to algebraic methods was blocked. The solver of the third case study, Solver #6, made some progress through the use of charting and became able to use a letter to represent a varying quantity, such as the number of miles traveled. She was able to describe the arithmetic process that she had used to create her charts by using algebraic symbolism, such as $(X \bullet .16) + 14$, but this phrase still represented a process to her.

The solvers of the last two case studies were able to attain the levels of structural abstraction or structural awareness. These solvers held or were able to develop robust conceptions of variable and equality. Significantly, the attainment of the level of structural awareness allowed the solver to view mathematical statements such as (X • .10) + 25 as both a process and an object. Being able to cope with the process/object duality (Sfard & Linchevski, 1994) made a transition to algebra possible for these solvers.

Thus, it is apparent that it is desirable for learners to attain high levels of reflective abstraction according to Cifarelli's constructs and to attain reification. In this regard, Sfard (1991) pointed out that while mathematical concepts can be conceived of operationally and structurally, and while, in many cases, an operational understanding will be satisfactory for one's purposes, operational concepts are limiting because they force one to think in terms of processes, which are detail oriented. This is a distinct disadvantage if one is working on a very complex problem. It is precisely under such circumstances that a structural viewpoint is useful. By being able to grasp complicated processes as static objects or quantities that can be manipulated and compared, one greatly reduces the cognitive strain induced by a purely operational approach. Sfard's observation is born out in Cifarelli's constructs because solvers who can attain the higher levels of reflective abstraction become progressively more flexible in their thinking and are thus able to confront a wider array of problems.

It seems that solvers must attain higher levels of reflective abstraction of they are to become able to think of processes as objects. This is why it is so crucial for students to be engaged in meaningful activities in the classroom. Next, one must consider what types of activities will encourage the transition from arithmetic to algebra to occur? Solvers need to work on problems that can be solved reasonably by arithmetic methods as well as algebraic methods. The results of the present study indicated that charting was a potentially productive means of encouraging the reflective abstraction that is

necessary for the transition to occur. In the case studies, the use of situation-specific imagery often led students to abandon purely arithmetic methods in favor of organizing their activity with a chart. Charts helped students to develop their understanding of variable and provided students with an opportunity to symbolize what they had done to create the chart. Getting students to reflect back on their own activity is crucial. The solvers constructed a transition from arithmetic to algebra by mathematizing their activity. This mathematization often occurred in layers--first, the solver attempted an arithmetic solution, next, the solver used a chart, and finally, the solver became able to symbolize the activity that created the chart. While the use of charts did not guarantee that a solver would successfully complete a transition to algebra, the appearance of charting served as evidence that meaningful activity was occurring. Further work on problems of this type would allow solvers to reflect on what was done and would provide them with the opportunity to reorganize their conceptual knowledge. Problem solving of this type, while time consuming, helps students to develop understandings of mathematical concepts that will be useful to them in the classroom and beyond.

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